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Erratum

Volume **25**, No. 3 (1973), in the article, "The Characterization of Finite Groups Whose Sylow 2-Subgroups Are of Type $L_3(q)$, q Even," by Michael J. Collins, pp. 490–512.

Page 508, the Statement of Lemma 8.13 should read:

$$|G| = (q^2 + 1) \cdot |N_G(B)|.$$

Pages 510–511, Section 9, The Proof of Theorem C: Parts of this section were omitted. The complete section is given below. For references see the original paper.

9. The Proof of Theorem C

The proof of our main result, Theorem B, now complete, we conclude the paper by proving Theorem C which gives a more precise description of the groups in question. In the course of doing so, we shall obtain Theorem 9.2 which is a more general result on irreducible modules for $SL(2, q)$ that is useful in studying groups with Sylow 2-subgroups of class 2.

If G is an insoluble group having Sylow 2-subgroups of type $L_3(q)$, then we have determined the simple chief factors. In case (ii) of Theorem B, we have

$$O''(G)/O(G) \cong L_3(q);$$

thus $G/O(G)$ is isomorphic to a subgroup of the automorphism group of $L_3(q)$. Since $\mathrm{Aut}(L_3(q))$ has a subgroup of index 2 isomorphic to $P\Gamma L(3, q)$ [9], Theorem C holds in this case.

We now consider case (iii) of Theorem B. We assume that q is fixed, and let S, A, and Z be as in previous sections. To obtain the conclusions of Theorem C, we may certainly suppose that $O(G) = 1$ and, without loss, that A is normal in G. Although we previously assumed that $q \geqslant 8$, it is clear that the following holds for $q = 4$ also; indeed, the corresponding result for that case is obtained in [5].

Hypothesis 9.1. *G is a group having a normal elementary abelian 2-subgroup A of order q^2 where $q = 2^n \geqslant 4$. The following hold:*

(a) *G has a subgroup X such that $G = AX$ and $A \cap X = 1$;*

(b) *$O'(X)$ is isomorphic to $SL(2, q)$;*

(c) $A = C_G(A)$; *and*

(d) *if T is a fixed Sylow 2-subgroup of X, then AT is isomorphic to S (and is so identified).*

We shall first determine the action of $O'(X)$ on A, but do so under a weaker hypothesis.

Suppose that V is an elementary abelian 2-group which affords a faithful module for $SL(2, q)$ where $q = 2^n$. We shall say that V is a *standard module* for $SL(2, q)$ if V has rank $2n$ and may be identified with the additive group of a vector space of dimension 2 over the field $GF(q)$ on which $SL(2, q)$ acts naturally.

THEOREM 9.2. *Let H be a finite group with Sylow 2-subgroups of class 2. Suppose that V is a normal elementary abelian 2-subgroup of H such that*

(a) $V = C_H(V)$,

(b) *H/V is isomorphic to $SL(2, q)$ where $q = 2^n \geqslant 4$, and*

(c) *H/V acts irreducible on V.*

Then V is a standard module for H/V.

Proof. We require information about the absolutely irreducible representations of $SL(2, q)$ in characteristic 2. There are q inequivalent absolutely irreducible representations of $SL(2, q)$. Let M_1 be the natural 2-dimensional module for $SL(2, q)$, and let $M_1, \ldots, M_n$ be the set of algebraic conjugates of M_1 induced by the field automorphisms of $GF(q)$. These are the *fundamental* modules. Then $GF(q)$ is a splitting field for $SL(2, q)$ and a full set of inequivalent absolutely irreducible representations for $SL(2, q)$ in characteristic 2 are the trivial representation and the representations afforded by the tensor products

$$M_{i_1} \otimes \cdots \otimes M_{i_r}, \qquad i_1 < i_2 < \cdots < i_r,$$

for $1 \leqslant r \leqslant n$. This result is well-known; it is sufficient to show that the modules in question are irreducible and nonisomorphic since their number is correct. It follows that $GF(q)$ is a splitting field for $SL(2, q)$. A detailed proof appears in [10].

Let ω be a primitive $(q - 1)$ root of unity in $GF(q)$, and put

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Let $\{e_1, e_2\}$ be the canonical basis for M_1 and (by abuse of notation) let $\{e_1, e_2\}$ also denote the corresponding basis for each M_i; if M_i is obtained from M_1 by applying the field automorphism α to $GF(q)$, then x acts on M_i by mapping

$$e_1 \to e_1 + e_2 \qquad \text{and} \qquad e_2 \to e_2$$

and y by

$$e_1 \to e_1 + \omega^\alpha e_2 \qquad \text{and} \qquad e_2 \to e_2 .$$

Now if $M = M_{i_1} \otimes \cdots \otimes M_{i_r}$ where $r \geqslant 2$, it is easily verified that the coefficient of $e_2 \otimes e_2 \otimes e_1 \otimes \cdots \otimes e_1$ in

$$(e_1 \otimes \cdots \otimes e_1)(1 - x)(1 - y)$$

is $(\omega^\beta + \omega^\gamma)$ where β and γ are the field automorphisms which give M_{i_1} and M_{i_2}. Since $\beta \neq \gamma$ and ω is primitive, $\omega^\beta \neq \omega^\gamma$. Hence

$$M(1 - x)(1 - y) \neq 0.$$

Under our hypothesis, if P is a Sylow 2-subgroup of H, then $[V, P, P] = 1$ since P has class 2. With bars denoting images in $\bar{H} = H/V$, this is equivalent to the linear condition

$$V(1 - \bar{g})(1 - \bar{h}) = 0 \qquad \text{for all } g, h \in P.$$

Let $F = GF(q)$. Considering V as a $GF(2)$-vector space and putting $V^* = V \otimes F$, for each composition factor N of V^* as an $F\bar{H}$-module and for all $g, h \in P$, we have

$$N(1 - \bar{g})(1 - \bar{h}) = 0.$$

We may choose g and h so that $\bar{g}$ and $\bar{h}$ may be identified with x and y above. Thus N cannot be isomorphic to any module $M_{i_1} \otimes \cdots \otimes M_{i_r}$ with $r \geqslant 2$ so that every composition factor of V^* is either trivial or a fundamental module. Since $\bar{H}$ acts irreducibly on V and F is separable over $GF(2)$, V^* is completely reducible as an $F\bar{H}$-module [3, Corollary 69.9]. Also, since the character of the representation afforded by V^* lies in $GF(2)$, each fundamental module appears as a composition factor with the same multiplicity e. Thus if M_0 is the trivial module for $SL(2, q)$ over F and f is its multiplicity in V^*, then

$$V^* \cong fM_0 \oplus e(M_1 \oplus \cdots \oplus M_n).$$

Now, if U is a standard module for $SL(2, q)$,

$$U \otimes F \cong M_1 \otimes \cdots \otimes M_n .$$

Hence, if L is a trivial module for $SL(2, q)$ over $GF(2)$,

$$(fL \oplus eU) \otimes F \cong V^* = V \otimes F.$$

By a result of Noether and Deuring [3, Theorem 29.11],

$$V \cong fL \oplus eU.$$

Since V is irreducible, U and V are isomorphic.

COROLLARY. *Under Hypothesis* 9.1, *A is a standard module for $O'(X)$.*

Proof. Considering A as an $O'(X)$-module, some composition factor A_1 must afford a faithful representation. A Sylow 2-subgroup of the natural semidirect product $A_1 \cdot O'(X)$ has class 2; hence, by Theorem 9.2, A_1 is a standard module. Thus $|A_1| = 2^{2n}$ so that $A_1 = A$.

We need the following for the final identification.

PROPOSITION 9.3. *Let $G = GL(3, q)$, given as a group of matrices, for $q = 2^n \geqslant 4$. Let A be the subgroup of G consisting of matrices of the form*

$$\begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Let $\Gamma = P\Gamma L(3, q)$ and let A also denote its own image in Γ. Then $N_\Gamma(A)$ is a group satisfying Hypothesis 9.1, *being a split extension of A by a group isomorphic to $\Gamma L(2, q)$.*

Proof. It is readily verified that $N_G(A)$ consists of the nonsingular matrices of the form

$$\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}$$

and that $C_G(A) = A \cdot \Lambda$ where Λ is the group of nonzero scalar matrices. The conclusion is now immediate.

The completion of the proof of Theorem C is now essentially reduced to a question of uniqueness; we shall show that a group satisfying Hypothesis 9.1 is isomorphic to a subgroup of $N_\Gamma(A)$ of Proposition 9.3.

Fix A as in Hypothesis 9.1, and let N be the subgroup of Aut(A) corresponding to the group $\Gamma L(2, q)$ of Proposition 9.3. Let M be the subgroup of N corresponding to the subgroup $SL(2, q)$ of $\Gamma L(2, q)$ and L that corresponding to $GL(2, q)$. Then, by the Corollary to Theorem 9.2, with X as in Hypothesis 9.1, we may identify $O'(X)$ with M. Let X also denote the subgroup of Aut(A) to which it corresponds.

We have, by the known structure of X, that

$$O(X) = C_X(O'(X)).$$

Since A is an irreducible $O'(X)$-module, $O(X)$ is isomorphic to a subgroup of the multiplicative group of a field by Schur's lemma, and so is cyclic. Since $O(X) \subseteq C_X(T)$, both $Z(=C_A(T))$ and A/Z admit $O(X)$. Hence $O(X)$ has order dividing $(q-1)$. By the same argument, $C_{\mathrm{Aut}(A)}(M)$ is also cyclic; hence

$$O(X) \subseteq Z(L) = C_{\mathrm{Aut}(\mathrm{A})}(M),$$

the latter equality since identifying T with a subgroup of M shows that Z admits $C_{\mathrm{Aut}(A)}(M)$, while $|Z(L)| = q - 1$.

Let x be an element of X outside $O''(X)$. Since the automorphism group of $SL(2, q)$ is generated by inner automorphisms and field automorphisms, we can find an element $y \in N$ such that xy centralizes M. Hence $xy \in Z(L)$ so that, in particular, $x \in N$. Thus $X \subseteq N$. Now the semidirect product AN is isomorphic to $N_\Gamma(A)$ which is a subgroup of $P\Gamma L(3, q)$. Thus Theorem C is established.

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